# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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## ON A CLASS OF EXACT SOLUTIONS OF THE EQUATIONS OF MOTION OF A VISCOUS FLUID

By V. I. Yatseyev

Translation

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# ON A CLASS OF EXACT SOLUTIONS OF THE EQUATIONS OF MOTION OF A VISCOUS FLUID*

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The general solution is obtained herein of the equations of motion of a viscous fluid in which the velocity field is inversely proportional to the distance from a certain point. Some particular cases of such motion are investigated.

1. The motion of a viscous fluid with velocity field and pressure in spherical coordinates can be given by the following expressions:

$$v_r = \frac{F(\theta)}{r} \qquad v_\theta = \frac{f(\theta)}{r} \qquad v_\varphi = 0 \qquad \frac{p}{\rho} = \frac{g(\theta)}{r^2} \tag{1}$$

A particular solution of the equations of Navier-Stokes for this case was obtained by Landau (reference 1). In the present paper a general solution is given of the equations of Navier-Stokes for the motion of the class under consideration.

Substituting expressions (1) in the equations of Navier-Stokes and in the equation of continuity yields the following system:

$$F^2 + f^2 - fF' + 2g - \nu\left[F'' + F' \cot\theta - 2f' - 2F - 2f \cot\theta\right] = 0 \tag{2}$$

$$ff' + g' - \nu\left[f'' + f' \cot\theta + 2F' - f\left(1 + \cot^2\theta\right)\right] = 0 \tag{3}$$

$$F + f' + f \cot\theta = 0 \tag{4}$$

Determining $F$ from equation (4) and substituting in equations (2) and (3) give

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$$f'^2 + ff'' + 3ff' \cot\theta + 2g - \nu\left[f''' + 2f''\cot\theta - f'\left(2 + \cot^2\theta\right) + f\cot\theta\left(1 + \cot^2\theta\right)\right] = 0 \tag{5}$$

$$ff' + g' + \nu\left[f'' + f'\cot\theta - f\left(1 + \cot^2\theta\right)\right] = 0 \tag{6}$$

Differentiating expression (6)

$$f'^2 + ff'' + g'' + \nu\left[f''' + f''\cot\theta - 2f'\left(1 + \cot^2\theta\right) + 2f\cot\theta\left(1 + \cot^2\theta\right)\right] = 0 \tag{7}$$

Eliminating the nonlinear terms $f'^2$, $ff''$, and $ff'$ from equation (5) with the aid of equations (6) and (7) yields a linear equation in the function $g + 2\nu f'$:

$$(g + 2\nu f')'' + 3\cot\theta\,(g + 2\nu f')' - 2\,(g + 2\nu f') = 0 \tag{8}$$

the general solution of which is in the form

$$g + 2\nu f' = 2\nu^2\,\frac{b\cos\theta - a}{\sin^2\theta} \tag{9}$$

where $2\nu^2 a$ and $2\nu^2 b$ are constants of integration.

Integrating equation (6)

$$f^2 + 2g + 2\nu\,(f' + f\cot\theta) = -\,2\nu^2 c \tag{10}$$

where $2\nu^2 c$ is the constant of integration.

The function $g(\theta)$ is eliminated from equations (9) and (10) to give an equation of the Riccati type for the function f:[1]

$$f' = \frac{1}{2\nu}f^2 + f\cot\theta + 2\nu\left(\frac{b\cos\theta - a}{\sin^2\theta} + \frac{c}{2}\right) \tag{11}$$

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[1]After sending the manuscript to press the author obtained from L. D. Landau a communication on the work of N. Slezkin (reference 2) in which he arrived at the same equation by a different method.

The substitution

$$f = -2\nu\chi'(\theta)/\chi(\theta) \tag{12}$$

reduces equation (11) to the linear equation:

$$\chi'' - \chi' \cot\theta + \left(\frac{b\cos\theta - a}{\sin^2\theta} + \frac{c}{2}\right)\chi = 0 \tag{13}$$

which by the substitution

$$z = \cos^2(\theta/2) \tag{14}$$

is transformed into an equation of the Fuchsian type:

$$\frac{d^2\chi}{dz^2} - \frac{a + b - 2(b + c)z + 2cz^2}{4z^2(z - 1)}\chi = 0 \tag{15}$$

The usual computations (reference 3), which are omitted herein, give the general solution of equation (15) as:

$$\begin{aligned}\chi(\theta) = \left(\cos\frac{\theta}{2}\right)^{\gamma}\left(\sin\frac{\theta}{2}\right)^{1+\alpha+\beta-\gamma}\Bigg\{&c_1F\left(\alpha,\beta,\gamma,\ \cos^2\frac{\theta}{2}\right) + \\ &c_2F\left(\alpha + 1 - \gamma,\beta + 1 - \gamma,2 - \gamma,\ \cos^2\frac{\theta}{2}\right)\Bigg\}\end{aligned} \tag{16}$$

where the parameters of the hypergeometric function $\alpha,\beta,\gamma$ (which can also have complex values) are connected with the constants of integration $a,b,c$ by the formulas:

$$\left.\begin{aligned} a &= \gamma^2 - (1 + \alpha + \beta)\gamma + \frac{(\alpha + \beta)^2}{2} - \frac{1}{2} \\ b &= (\alpha + \beta - 1)\gamma - \frac{(\alpha + \beta)}{2} + \frac{1}{2} \\ c &= \frac{(\alpha - \beta)^2 - 1}{2} \end{aligned}\right\} \tag{17}$$

Formulas (4), (9), (16), and (17) give the general solution, depending on the four constants a, b, c, and $A = c_2/c_1$, of the Navier-Stokes equations for the class of motion of a viscous fluid under consideration. The constants of integration a, b, and c are expressed in terms of the corresponding tensor components of the density of the momentum transfer:

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \rho\nu\left(\frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i}\right) \tag{18}$$

Carrying out the computations

$$\left.\begin{aligned} \Pi_{\varphi\varphi} &= \frac{2\nu^2\rho}{r^2}\left(\frac{b\cos\theta - a}{\sin^2\theta}\right) \\ \Pi_{\theta\theta} &= \frac{2\nu^2\rho}{r^2}\left(\frac{a - b\cos\theta}{\sin^2\theta} - \frac{c}{2}\right) \\ \Pi_{r\theta} &= \frac{2\nu^2\rho}{r^2}\left(\frac{c\cos\theta - b}{\sin^2\theta}\right) \end{aligned}\right\} \tag{19}$$

The streamlines are determined by the equation:

$$dr/v_r = rd\theta/v_\theta \tag{20}$$

the integration of which gives

$$\text{const}/r = f\sin\theta \tag{21}$$

2. Attention is now given to two particular examples for which the equation of Fuchs degenerates.

(a) Equation (15) has only one regular singular point, $z = \infty$. In this case

$$a = b = c = 0 \tag{22}$$

and therefore by equations (19)

$$\Pi_{\varphi\varphi} = \Pi_{\theta\theta} = \Pi_{r\theta} = 0 \tag{23}$$

The particular solution of equation (15)

$$\chi(\theta) = 2z - 1 - A \tag{24}$$

leads by formulas (4), (9), and (12) to the solution found by Landau:

$$F(\theta) = 2\nu\left[\frac{A^2 - 1}{(A - \cos\theta)^2} - 1\right] \qquad f(\theta) = \frac{2\nu \sin\theta}{\cos\theta - A}$$

$$g(\theta) = 4\nu^2 \frac{1 - A\cos\theta}{(\cos\theta - A)^2} \tag{25}$$

This solution is analogous to the problem of a stream flowing out of the end of a thin pipe into a region filled with the same fluid. It is the only regular solution for all values of the angle $\theta$.

(b) Equation (15) has only two regular points $z = 0$ and $z = \infty$. In this case it follows from equation (15) that

$$a = b = c \neq 0 \tag{26}$$

and equation (11) becomes Euler's equation

$$2z^2 \left(d^2\chi/dz^2\right) - a\chi = 0 \tag{27}$$

the general solutions of which are

$$\left.\begin{aligned} \chi(\theta) &= e^{x/2} \cosh (nx + A) \quad \text{for } a > -1/2 \\ \chi(\theta) &= e^{x/2} \cos (nx + A) \quad \text{for } a < -1/2 \\ \chi(\theta) &= e^{x/2} (1 + Ax) \quad \text{for } a = -1/2 \end{aligned}\right\} \tag{28}$$

Correspondingly, the following equations are obtained for the function $f(\theta)$:

$$f = 2\nu \frac{\sin\theta}{1 + \cos\theta}\left\{n \tanh (nx + A) + 1/2\right\} \quad \text{for } a > -1/2 \tag{29}$$

$$f = 2\nu \frac{\sin\theta}{1 + \cos\theta}\left\{\frac{1}{2} - n \tan (nx + A)\right\} \quad \text{for } a < -1/2 \tag{30}$$

$$f = 2\nu \frac{\sin\theta}{1 + \cos\theta}\left\{\frac{A}{1 + Ax} + 1/2\right\} \quad \text{for } a = -1/2 \tag{31}$$

where $x = \ln(1 + \cos\theta)$, $n = \frac{1}{2}\left|\sqrt{1 + 2a}\right|$. (For $a = 0$, $n = 1/2$ in equation (29) the solution of Landau is again obtained.)

For the solution of equation (29) by formula (4)

$$F(\theta) = -2\nu\left\{n \tanh(nx + A) + 1/2\right\} + 2\nu\frac{1 - \cos\theta}{1 + \cos\theta}\frac{n^2}{\cosh^2(nx + A)} \tag{32}$$

while $g(\theta)$ is determined by formula (9).

The equation of the streamlines is in the form

$$\mathrm{const}/r = (1 - \cos\theta)\left\{n \tanh(nx + A) + 1/2\right\} \tag{33}$$

where the values of the constants $n$ and $A$ are determined from the conditions

$$\left.\begin{aligned} f\left(\frac{\pi}{2}\right) &= 2\nu\ (n \tanh A + 1/2) \\ f\left(\frac{\pi}{2}\right) + F\left(\frac{\pi}{2}\right) &= 2\nu\frac{n^2}{\cosh^2 A} \end{aligned}\right\} \tag{34}$$

The obtained solution corresponds to the problem of the stream flowing from the half line $\theta = \pi$ into a region filled with the same fluid.

For solution (30), the parametric equation for the streamlines is in the form

$$\begin{aligned} \mathrm{const}/r &= (2 - e^x)\left[1/2 - n \tan(nx + A)\right] \\ \theta &= \arccos(e^x - 1) \end{aligned} \tag{35}$$

The function (35) for $\theta \to \pi$ is a strongly oscillating one. It can therefore be concluded that solution (30) has no physical sense.

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